

## Amortization Analysis (20 pts)

Suppose we have  $m$  heavy stone tablets each with a “0” on one side and “1” on the other side. We want to use these tablets as an incrementing counter representing  $m$  bits. Initially all  $m$  stone tablets start at 0. And each step, we will simply increment the counter.

For example, below is an example when  $m = 8$ .

|         |                 |
|---------|-----------------|
| Initial | 0 0 0 0 0 0 0 0 |
| Step 1  | 0 0 0 0 0 0 0 1 |
| Step 2  | 0 0 0 0 0 0 1 0 |
| Step 3  | 0 0 0 0 0 0 1 1 |
| Step 4  | 0 0 0 0 0 1 0 0 |
| ...     | 0 0 0 0 0 0 0 0 |

From initial to step 1, the 8<sup>th</sup> stone was flipped from 0 to 1

From step 1 to step 2, the 8<sup>th</sup> stone was flipped from 1 to 0 and the 7<sup>th</sup> stone was flipped from 0 to 1.

From step 2 to step 3, the 8<sup>th</sup> stone was flipped from 0 to 1.

From step 3 to step 4, the 6<sup>th</sup> stone was flipped from 0 to 1, and the 7<sup>th</sup> and 8<sup>th</sup> stone was flipped from 1 to 0.

And so on...

Assume the flipping stone from 0 to 1 and 1 to 0 costs the same, and  $m$  is sufficiently large for a sequence of  $n$  increments.

Q1. In a sequence of  $n$  increments, what is the worst-case cost for an increment in terms of  $n$ ?

Q2. In a sequence of  $n$  increments, what is the amortized cost per increment? Use accounting method.

In your description, specify what should be the charge for 0→1 flip and 1→0 flip and provide the reasoning.

Q3. Potential method?

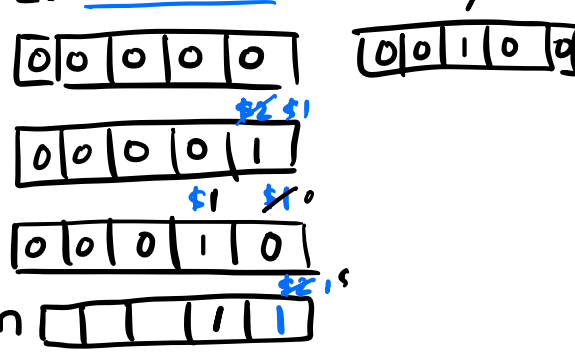
### 1. aggregate method

0<sup>th</sup> bit : flips every time =  $n$   
 1<sup>st</sup> bit : flips every 2<sup>nd</sup> increment  
 2<sup>nd</sup> bit : flips every 4<sup>th</sup> increment  
 3<sup>rd</sup> bit : flips every 8<sup>th</sup> increment

$$n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \dots = \boxed{n} + \boxed{\frac{n}{2}} = \boxed{2n}$$

$n$  increments  $\rightarrow 2n$   
 per cost  $\rightarrow 2 \Rightarrow O(1)$

### 2. Banker's method



### 3. Potential method.

Potential function  $B_i = \# \text{ of } 1\text{'s in the counter after } i\text{th operation}$

Suppose  $i$ th operation resets  $(t_i)$  bits  
 $1 \rightarrow 0$

0111  $\rightarrow$  1000  
 $t_i = 3$

Actual cost  $c_i = t_i + 1$

0001  $\xrightarrow{t_i=1}$   
 0010  $\xrightarrow{t_i=0}$   
 0011  $\xrightarrow{t_i=0}$

$$B_i = B_{i-1} - t_i + 1$$

$$B_i - B_{i-1} = B_{i-1} - t_i + 1 - B_{i-1} = 1 - t_i$$

potential energy  
change by  
 $i$ th func.

$$a_i = c_i + 1 - t_i = t_i + 1 + 1 - t_i = 2$$