Amortization Analysis (20 pts)

Suppose we have m heavy stone tablets each with a "0" on one side and "1" on the other side. We want to use these tablets as an incrementing counter representing m bits. Initially all m stone tablets start at 0. And each step, we will simply increment the counter.

For example, below is an example when m = 8.

Initial	0 0 0 0 0 0 0 0
Step 1	0 0 0 0 0 0 0 1
Step 2	0 0 0 0 0 0 1 0
Step 3	0 0 0 0 0 0 1 1
Step 4	0 0 0 0 0 1 0 0
…	0 0 0 0 0 1 0 1

0 0 0 0 0 1 1 0

0 0 0 0 0 1 1 1

0 0 0 0 1 0 0 0

From initial to step 1, the 8th stone was flipped from 0 to 1
From step 1 to step 2, the 8th stone was flipped from 1 to 0 and the 7th stone was flipped from 0 to 1.
From step 2 to step 3, the 8th stone was flipped from 0 to 1.
From step 3 to step 4, the 6th stone was flipped from 0 to 1, and the 7th and 8th stone was flipped from 1 to 0.
And so on…

Assume the flipping stone from 0 to 1 and 1 to 0 costs the same, and m is sufficiently large for a sequence of n increments.

Q1. ***In a sequence of n increments, what is the worst-case cost for an increment in terms of n?***

Q2. ***In a sequence of n increments, what is the amortized cost per increment? Use accounting method.*** In your description, specify what should be the charge for 0->1 flip and 1->0 flip and provide the reasoning.

Q3. Potential method?

1. aggregate method

0th bit : flips ~~everytime~~ increment = n

1th bit : flips every 2nd increment

2nd bit : flips every 4th increment

3rd bit : flips every 8th increment

$n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots = \underset{n}{\square} + \underset{n}{\square} = \boxed{2}n$

n increments → 2n

per incr cost → 2 ⇒ O(1)

2. Bankers method

| 0 | 0 | 0 | 0 | 0 |

| 0 | 0 | 0 | 0 | 1 | ~~\$2~~ \$1

| 0 | 0 | 0 | 1 | 0 | \$1 ~~\$1~~ 0

| | | | 1 | 1 | ~~\$2~~ 1

| 0 | 0 | 1 | 0 | 0 |

3. Potential method.

Potential function B_i = # of 1's in the counter after ith operation

$0111 \to 1000$
$t_i = 3$

Suppose ith operation resets t_i bits
$1 \to 0$

0001
0010 $t_1=1$
0011 $t_2=0$

Actual cost $c_i = t_i + 1$

$B_i = B_{i-1} - t_i + 1$

$B_i - \boxed{B_{i-1}} = B_{i-1} - t_i + 1 - \cancel{B_{i-1}} = \boxed{1 - t_i}$

potential energy change by ith inc.

$\boxed{a_i} = c_i + 1 - t_i = \cancel{t_i} + 1 + 1 - \cancel{t_i} = \boxed{2}$